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Black-hole back-reaction—a toy model

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Abstract
Using a kink of arbitrary shape as a toy model for a black hole (horizon), we study the back-reaction of the evaporation process and find that the horizon is always pushed back (as in the gravitational case). The associated heat capacity and entropy variation, on the other hand, can be positive or negative, depending on the parameters.

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1. Motivation

Our standard picture of black holes includes several rather odd features, which might give us some hints about quantum gravity. Due to the Hawking effect [1], black holes evaporate by emitting thermal radiation with the temperature being inversely proportional to the mass[3]. Hence they constantly lose energy via this evaporation process, and thereby their horizon shrinks. As a result, their heat capacity is negative, i.e. they get hotter by losing energy (see footnote 3). Finally, the generalization of the second law of thermodynamics to black holes yields an entropy, which is proportional to the surface area (instead of the volume, for example).

In order to understand the oddity of these features, it is useful to construct toy models which reproduce some of the relevant properties of black holes, but are still simple enough to do the calculations (which we cannot do in quantum gravity). In the following, we propose a very simple toy model for black-hole evaporation and study these issues; see [2].

[3] For simplicity, we consider the asymptotically flat Schwarzschild metric only, i.e. a black hole without charge and angular momentum. Otherwise, we would have to distinguish different heat capacities in analogy to c_p and c_V in thermodynamics.

2. Kink dynamics

Let us start by briefly reviewing some of the basic properties of kinks, see, e.g., [3, 4]. To this end, we consider a scalar field ψ in 1+1 dimensions ($\hbar = 1$):

$$\mathcal{L}_\psi = \frac{1}{2}\left(\dot{\psi}^2 - c_\psi^2[\partial_x\psi]^2\right) - V(\psi), \tag{1}$$

where the potential is supposed to be non-negative $V(\psi) \geqslant 0$ such that its zeros ψ_n denote the degenerate classical ground states $V(\psi_n) = 0$. Furthermore, we assume a very stiff potential which allows us to approximate the quantum field ψ by a classical solution (plus small quantum corrections). For any (non-negative) potential $V(\psi)$ with more than one ground state, there exists at least one stable topological defect in the form of a static kink solution $\psi_{\text{kink}}(x)$ which connects two adjoining zeros $\psi_{\text{kink}}(x \to -\infty) = \psi_n$ and $\psi_{\text{kink}}(x \to +\infty) = \psi_{n+1}$ or vice versa (anti-kink). These kinks are finite-energy solutions of the classical field equations and their stability (even in the non-static case) is protected by topology. The static kink solution $\psi_{\text{kink}}(x)$ satisfies $c_\psi^2[\partial_x\psi_{\text{kink}}]^2 = 2V(\psi_{\text{kink}})$ (virial theorem) and hence obeys the implicit relation

$$\psi_{\text{kink}} = \psi_{\text{kink}}(x) \leftrightarrow x(\psi) = \pm\int^\psi \frac{\mathrm{d}\tilde{\psi}}{\sqrt{2V(\tilde{\psi})}}. \tag{2}$$

A moving kink can be described in terms of the collective coordinate $X[t]$ determining the kink position. Accordingly, we split up the total quantum field ψ into a classical kink solution with a quantized position operator $X[t]$ (in the non-relativistic limit $\dot{X}^2 \ll c_\psi^2$) plus quantum corrections via

$$\psi(t, x) = \psi_{\text{kink}}(x - X[t]) + \delta\psi(t, x). \tag{3}$$

In order to avoid double counting [3, 4], we demand $\delta\psi \perp \partial_x\psi_{\text{kink}}$ (see below). The insertion of this (non-relativistic) ansatz into the action (1) yields

$$L_\psi = \frac{1}{2}M_{\text{eff}}\dot{X}^2 - \frac{1}{2}\int \mathrm{d}x\,\delta\psi\left(\partial_t^2 - c_\psi^2\partial_x^2 - V''(\psi_{\text{kink}})\right)\delta\psi + \mathcal{O}(\delta\psi^3), \tag{4}$$

plus an irrelevant constant. The differential operator on the rhs is self-adjoint and non-negative (since the kink is stable). It possesses one zero mode which is just given by $\partial_x\psi_{\text{kink}}$ and corresponds to a translation of the kink (i.e. to $X[t]$). Since this mode is excluded from the Hilbert space $\delta\psi \perp \partial_x\psi_{\text{kink}}$, all other modes have positive energies (stability).

3. Hawking radiation

Taking the kink (with a well-defined kinetic energy) as a model for the black hole (horizon), we may now add the ingredient of Hawking radiation. To this end, we introduce a light (massless) scalar quantum field ϕ coupled to the classical kink solution according to

$$\mathcal{L}_\phi = \frac{1}{2}\left([\partial_t\phi + g\psi\partial_x\phi]^2 - c_\phi^2[\partial_x\phi]^2\right). \tag{5}$$

In this way, the light field ϕ propagates in this kink background in the same way as in a curved spacetime with the effective metric [5]

$$\mathrm{d}s^2 = \left(c_\phi^2 - v^2\right)\mathrm{d}t^2 - 2v\,\mathrm{d}t\,\mathrm{d}x - \mathrm{d}x^2, \tag{6}$$

where $v = g\psi$ denotes the local frame-dragging velocity. If this velocity v exceeds the propagation speed c_ϕ, we get the analogue of a black-hole horizon. Exploiting the analogy to gravity, we may introduce the pseudo energy–momentum tensor of the field ϕ:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{A}_\phi}{\delta g^{\mu\nu}} = (\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}g_{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi). \tag{7}$$

Note that this quantity is not conserved itself in general $\partial_\mu T^\mu_\nu \neq 0$; see the appendix. Instead, the analogue of the covariant energy–momentum balance,

$$\nabla_\mu T^\mu_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}T^\mu_\nu\right) - \frac{1}{2}T^{\alpha\beta}\partial_\nu g_{\alpha\beta} = 0, \tag{8}$$

contains a term $T^{\alpha\beta}\partial_\nu g_{\alpha\beta}/2$ which describes the exchange of energy/momentum between the ϕ-field and the gravitational field (in curved spacetimes) or the ψ-field (in our case); see the appendix. Fortunately, in 1+1 dimensions, the expectation value of T^μ_ν determining the Hawking radiation can be calculated analytically via the trace anomaly [6]. For example, the mixed component $\langle T^0_1\rangle$ which will become relevant reads (for an arbitrary massless scalar field)

$$\langle T^0_1\rangle = \frac{4vc_\phi(\kappa^2 - [v']^2 - \gamma v v'') - \kappa^2(c_\phi + v)^2}{48\pi c_\phi^3\gamma^2}, \tag{9}$$

with $\gamma = 1 - v^2/c_\phi^2$ and the effective surface gravity κ determining the Hawking temperature is given by the velocity gradient at the horizon [5][4]:

$$T_{\text{Hawking}} = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)_{v^2=c_\phi^2}. \tag{10}$$

4. Back-reaction

Now let us estimate the back-reaction of the emitted Hawking radiation onto the kink. Variation of total action in equations (4) plus (5) w.r.t. the collective coordinate X yields

$$M_{\text{eff}}\ddot{X} = g\int \mathrm{d}x\, T^0_1\partial_x\psi_{\text{kink}} + \mathcal{O}(\delta\psi^2), \tag{11}$$

i.e. the acceleration of the kink induced by the evaporation process is determined by the overlap between the translational zero-mode $\partial_x\psi_{\text{kink}}$ and the above component of the pseudo energy–momentum tensor (9). Inserting the expectation value, we see that $v'\langle T^0_1\rangle$ is a total differential and hence we get

$$M_{\text{eff}}\langle\ddot{X}\rangle = -\left.\frac{2v^2(v')^2 + c_\phi\kappa^2(v - c_\phi)}{48\pi c_\phi^3\gamma}\right|_{-\infty}^{+\infty} = \left.\frac{\kappa^2}{48\pi(v + c_\phi)}\right|_{-\infty}^{+\infty}. \tag{12}$$

Consequently, the kink is pushed back by the emitted Hawking radiation.

This result can also be understood via the energy–momentum balance. Far away from the kink, we may estimate the energy–momentum tensor of the outgoing Hawking radiation and the associated in-falling partner particles via the geometric-optics approximation by replacing $\dot{\phi} \to \Omega$ and $\phi' \to k$ and using the dispersion relation $(\Omega + vk)^2 = c_\phi^2k^2$. For the component (9), this yields

$$T^0_1 = [\partial_t\phi + v\partial_x\phi]\partial_x\phi \to -\frac{c_\phi\Omega^2}{(c_\phi + v)^2}. \tag{13}$$

Identifying $\Omega \sim \kappa$, this expression coincides with equation (9) far away from the kink, where v'' and v' approach zero. The above quantity is the momentum density, which should not be confused with the energy flux density

$$T^1_0 = \dot{\phi}\left[v\dot{\phi} + \left(v^2 - c_\phi^2\right)\phi'\right] \to c_\phi\Omega^2, \tag{14}$$

[4] Note that calculating the expectation value $\langle T^\mu_\nu\rangle$ via the trace anomaly does not tell us that the radiation is thermal—this property can be inferred from a direct derivation of the Bogoliubov coefficients; see, e.g., [2] and references therein.

which is constant. (For a static kink, this is even true exactly, i.e. without the geometric-optics approximation.) Finally, the energy density is given by

$$T_0^0 = \frac{1}{2}\left([\partial_t\phi]^2 + \left(c_\phi^2 - v^2\right)[\partial_x\phi]^2\right) \to \frac{c_\phi\Omega^2}{c_\phi + v}, \tag{15}$$

and within the geometric-optics approximation (i.e. neglecting the trace anomaly), one obtains the same expression for the pressure $T_1^1 = -T_0^0$. Therefore, the particles of the Hawking radiation carry away positive energy and momentum, but their in-falling partner particles act in the opposite way and carry negative energy and momentum. Consequently, the energy flux is balanced, but the momentum flux is not. The pressure outside the horizon is positive and the pressure inside is negative, i.e. both contributions generate a force onto the kink in the same direction and hence the horizon is pushed back. Identifying $\Omega \sim \kappa$, the pressure difference yields the same force as in (12). Note that the asymptotic values of $\langle T_\nu^\mu\rangle$ can also be calculated directly from the Bogoliubov coefficients, i.e. without any reference to the trace anomaly and possible renormalization ambiguities (cf the appendix).

The expectation value (9) was evaluated in the Unruh state, which describes the evaporation process and thus contains outgoing thermal radiation, but no incoming particles. Adding an equal amount of incoming thermal radiation (e.g., by placing a mirror for the ϕ-field far away from the kink) would correspond to the (1+1 dimensional) Israel–Hartle–Hawking state. In this case, the relevant expectation value reads

$$\langle\hat{T}_1^0\rangle_{\rm IHH} = \frac{4vc_\phi(\kappa^2 - [v']^2 - \gamma v v'')}{48\pi c_\phi^3\gamma^2}. \tag{16}$$

In contrast to (9), this quantity is invariant under velocity reversal $v \to -v$, and hence the Israel–Hartle–Hawking state is regular across both (black and white holes) horizons $v = \pm c_\phi$. Nevertheless, the horizon is still pushed inwards:

$$M_{\rm eff}\langle\ddot{X}\rangle_{\rm IHH} = -\left.\frac{v^2(v')^2 - c_\phi^2\kappa^2}{24\pi c_\phi^3\gamma}\right|_{-\infty}^{+\infty}. \tag{17}$$

In gravity, the Israel–Hartle–Hawking state could describe a black hole surrounded by a large spherical mirror and would correspond to a stationary (though not necessarily stable) state of the total system (black hole plus radiation field). In contrast, the kink/horizon is still pushed back in our toy model, i.e. the Israel–Hartle–Hawking state does not correspond to an equilibrium state of the combined system (kink plus ϕ-field). This finding is related to the fact that the Lagrangian of our toy model in equations (1) and (5) is only invariant under time reversal if we simultaneously invert $\psi \to -\psi$, which does not leave the background kink solution unaffected, but turns it into an anti-kink (corresponding to a white-hole horizon $v \to -v$).

5. Conclusions

Using a kink of arbitrary shape as a toy model for a black hole (horizon), we found that the back-reaction of the evaporation process always tends to push the horizon inwards (i.e. the black hole to shrink, see also [7]). However, in contrast to real black holes in gravity, this shrinkage is not caused by energy conservation, but by momentum balance (and even persists for the Israel–Hartle–Hawking state).

Going a step further, one may also study the associated heat capacity and entropy variation of the kink [2]. It turns out that both could be positive or negative, depending on the chosen parameters.

Together with other analogues of black holes (see, e.g., [5, 7, 8]), our findings suggest that Hawking radiation and possibly the shrinkage of the horizon are quite robust phenomena, which are fairly independent of the Einstein equations. In contrast, understanding the origin of the black-hole entropy and heat capacity probably requires some aspects of the Einstein equations.

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Appendix A

Let us compare the pseudo energy–momentum tensor $T_{\mu\nu}$ of the field ϕ in equation (7) with the total energy–momentum tensor $\mathfrak{T}^\mu_\nu$ derived from the full Lagrangian $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi$ with respect to the Minkowski metric. Since $\mathcal{L}_\phi$ does not contain derivatives of the ψ-field, we get

$$\mathfrak{T}^\mu_\nu = \mathfrak{T}^\mu_\nu[\psi] + \mathfrak{T}^\mu_\nu[\phi] = \left[\frac{\partial \mathcal{L}_\psi}{\partial \psi_{,\mu}}\partial_\nu\psi - \delta^\mu_\nu \mathcal{L}_\psi\right] + \left[\frac{\partial \mathcal{L}_\phi}{\partial \phi_{,\mu}}\partial_\nu\phi - \delta^\mu_\nu \mathcal{L}_\phi\right]. \tag{A.1}$$

Obviously, the sum of both contributions must be conserved in a meaningful renormalization scheme $\partial_\mu\langle\mathfrak{T}^\mu_\nu\rangle = 0$, but each part is separately not conserved in general $\partial_\mu\langle\mathfrak{T}^\mu_\nu[\psi]\rangle = -\partial_\mu\langle\mathfrak{T}^\mu_\nu[\phi]\rangle \neq 0$, i.e. there will be an exchange of energy/momentum between the two fields, which is precisely the back-reaction force under consideration. Using the equations of motion for ψ and the assumption that ψ should be well approximated by a classical field, we get

$$\partial_\mu\langle\mathfrak{T}^\mu_\nu[\psi]\rangle = \left[\ddot\psi - c_\psi^2\partial_x^2\psi + V'(\psi)\right]\partial_\nu\psi = -g\,\langle[\partial_t\phi + g\psi\partial_x\phi]\partial_x\phi\rangle\,\partial_\nu\psi. \tag{A.2}$$

Since the expectation value on the rhs is just $\langle\mathfrak{T}^0_1[\phi]\rangle$, this equality is the analogue of equation (11) and shows that the energy/momentum exchange is determined by $\partial_\mu\langle\mathfrak{T}^\mu_\nu[\psi]\rangle = -\langle\mathfrak{T}^0_1[\phi]\rangle\partial_\nu v$.

Now, even though the two tensors $\mathfrak{T}_{\mu\nu}[\phi]$ and $T_{\mu\nu}$ are defined w.r.t. different metrics and hence cannot be compared in general $\mathfrak{T}_{\mu\nu}[\phi] \neq T_{\mu\nu}$, it turns out that the *mixed* components coincide $\mathfrak{T}^\mu_\nu[\phi] = T^\mu_\nu$. Hence, switching from the Minkowski metric to the effective metric $g_{\mu\nu}$ in equation (6), we find

$$\partial_\mu\langle T^\mu_\nu\rangle = \partial_\mu\langle\mathfrak{T}^\mu_\nu[\phi]\rangle = -\partial_\mu\langle\mathfrak{T}^\mu_\nu[\psi]\rangle = \langle\mathfrak{T}^0_1[\phi]\rangle\partial_\nu v = \langle T^0_1\rangle\partial_\nu v = \tfrac{1}{2}\langle T^{\alpha\beta}\rangle\partial_\nu g_{\alpha\beta}. \tag{A.3}$$

Thus, by exploiting the total energy–momentum conservation $\partial_\mu\langle\mathfrak{T}^\mu_\nu\rangle = 0$, we are able to *derive* the covariant balance law $\nabla_\mu\langle T^\mu_\nu\rangle = 0$ instead of just assuming it (note that $\sqrt{-g} = \text{const}$). This is a non-trivial statement because the validity of $\nabla_\mu\langle T^\mu_\nu\rangle = 0$ forces us to abandon the conformal invariance $\langle T^\mu_\mu\rangle = 0$ which leads to the trace anomaly [6]. Demanding $\nabla_\mu\langle T^\mu_\nu\rangle = 0$ is then a condition on the used renormalization scheme and is usually motivated via covariance arguments (whose applicability is not obvious in our model).

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